

Crossing numbers of periodic graphs

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Abstract

A graph is *periodic* if it can be obtained by joining identical pieces in a cyclic fashion. It is shown that the limit crossing number of a periodic graph is computable. This answers a question of Richter [1, Problem 4.2].

1 Introduction

The asymptotic behavior of the crossing number of periodic graphs, i.e., the graphs that can be obtained by joining identical pieces in a cyclic fashion, plays fundamental role in constructions of crossing-critical graphs [4] and in explaining certain phenomena. The first systematic treatment of this area is due to Pinontoan and Richter [5], who provided basic results and motivated several questions. In this paper we provide a simplified, yet equivalent setting for considering periodic graphs and answer a question of Pinontoan and Richter.

We allow graphs to have loops and parallel edges. A *tile* is a triple $T = (G, A, B)$, where G is a graph and $A = (a_1, a_2, \dots, a_k)$ and $B = (b_1, b_2, \dots, b_k)$

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are sequences of vertices of G of the same length k . Note that each vertex of G may appear several times in A and B . The length k of the sequences is called the *width* of the tile, and the graph G is the *underlying graph* of the tile. If $T_1 = (G_1, A_1, B_1)$ and $T_2 = (G_2, A_2, B_2)$ are tiles of the same width, we denote by $T_1 T_2$ the tile (G, A_1, B_2) , where G is the graph obtained from the disjoint union of G_1 and G_2 by adding, for $1 \leq i \leq k$, an edge between the i -th vertices of B_1 and A_2 . By $\odot(T_1)$, we denote the graph obtained from G_1 by adding edges between the i -th vertices of A_1 and B_1 for $i = 1, \dots, k$. If T is a tile, we define $T^1 = T$ and $T^n = T^{n-1}T$ for integers $n > 1$. We call the edges of $\odot(T_1^n)$ that belong to the copies of G_1 *internal*, and the edges between the copies *external*.

For a tile T , we would like to determine the asymptotic behavior of the crossing number of $\odot(T^n)$. Let us remark that this can significantly differ from the crossing number of T^n (compare e.g. the cylindrical $m \times n$ grid with the toroidal $m \times n$ grid). Let $c_n(T) = \text{cr}(\odot(T^n))$. Pinontoan and Richter proved in [5] that the limit

$$c(T) = \lim_{n \rightarrow \infty} \frac{c_n(T)}{n}$$

exists (they prove this only for tiles whose underlying graph is connected, however the claim holds for disconnected tiles as well, see Lemma 9). However, their proof gives no way to determine or to estimate the limit.¹ Our main result is a bound on the convergence rate of the limit, giving an algorithm to approximate $c(T)$ within arbitrary precision.

Theorem 1. *For every tile T and for every $\varepsilon > 0$, there exists a computable constant $N = O(1/\varepsilon^6)$ such that*

$$\left| \frac{c_t(T)}{t} - c(T) \right| \leq \varepsilon$$

for every integer $t \geq N$.

Therefore, to estimate the limit within given precision ε , it suffices to determine the constant N of Theorem 1 and compute the crossing number

¹The abstract of [5] erroneously states an estimate on the convergence rate of the limit, but the results of the paper do not give this, as confirmed in a private communication with the authors.

of $\odot(T^N)$, which can be done using the algorithm of Chimani et al. [2]. The resulting algorithm is exponential in a polynomial of $\frac{1}{\varepsilon}$.

It seems intuitively obvious that for large n , there is an optimal drawing of $\odot(T^n)$ exhibiting some kind of periodic behavior, and thus $c(T)$ is a rational number. Nevertheless, we were not able to prove this, and we propose the following conjecture.

Conjecture 1. *There exists a computable function f that assigns to every tile T a positive integer $f(T)$ such that the number $c(T)$ is rational with denominator at most $f(T)$.*

If this were true, our result would give an algorithm to determine $c(T)$ exactly, by choosing $\varepsilon = \frac{1}{2f^2(T)}$ and rounding the result to the nearest rational number with denominator at most $f(T)$. Note that $c(T)$ is not always integral, as shown by Pinontoan [3].

Let us remark that we define the cyclic composition $\odot(T^n)$ of copies of a tile $T = (G, A, B)$ in a way slightly different from the definition of Pinontoan and Richter [5]. They assume that the $2k$ vertices appearing in A and B are all distinct and that for $i, j = 1, \dots, k$, a_i and a_j are adjacent if and only if b_i and b_j are adjacent. Then, instead of adding the edges $a_i b_i$ joining consecutive copies of T , they identify a_i and b_i for $i = 1, \dots, k$. Our choice of a different definition turns out to be very convenient in our proofs, as we can treat the external edges added between the tiles specially.

It is easy to see that the same set of cyclic graphs can be constructed by both definitions. Given a tile T in our definition, we can add vertices a'_i and b'_i and edges $a_i a'_i$ and $b_i b'_i$ for $i = 1, \dots, k$, and after this change the construction of [5] gives the same result as our definition up to vertices of degree 2, which have no effect on the crossing number. On the other hand, given the cyclic composition H of n copies of T constructed as in [5], we can choose an arbitrary edge-cut in T separating A from B and split H on the copies of this edge-cut, resulting in n identical tiles which can be composed to form H using our definition of tile composition.

2 Connectivity and linkedness

Before starting with the main proofs, we show that it suffices to consider tiles that are connected and linked. We say that a tile is *connected* if its underlying graph is connected. A tile (G, A, B) of width k is *weakly linked* if

there exist pairwise edge-disjoint paths P_1, \dots, P_k in G and a permutation π of $[k]$ such that for $1 \leq i \leq k$, the path P_i starts at the i -th element of A and ends at the $\pi(i)$ -th element of B . A tile (G, A, B) of width k is *linked* if there exist pairwise edge-disjoint paths P_1, \dots, P_k in G such that for $1 \leq i \leq k$, the endvertices of P_i are the i -th elements of A and B .

Proposition 2. *For every tile $T = (G, A, B)$ of width k there exists a weakly linked tile T_0 of width $t \leq k$ such that for every $n \geq 1$, we have $\odot(T_0^n) = \odot(T^n)$.*

Proof. Let G' be the graph obtained from G by adding new vertices a and b and k edges between a and the elements of A and k edges between b and the elements of B . Let us first consider the case that there exists a minimal edge-cut $S = \{s_1, s_2, \dots, s_t\}$ in G' separating a from b with $t < k$. For $1 \leq i \leq t$, let u_i be the vertex incident with s_i belonging to the component of $G' - S$ containing a , if this vertex is different from a . If s_i is incident with a and the edge s_i joins a with the j -th element of A , then let u_i be the j -th element of B . Symetrically, let v_i be the other vertex of s_i if it is not equal to b , and let v_i be the corresponding element of A if s_i is incident with b . Let T_0 be the tile $(G'', (v_1, \dots, v_t), (u_1, \dots, u_t))$, where G'' is the graph obtained from $\odot(T)$ by removing the edges $u_1v_1, u_2v_2, \dots, u_tv_t$. It is easy to see that $\odot(T_0^n) = \odot(T^n)$ for every $n \geq 1$.

Since the width of T_0 is smaller than the width of T , we can perform this transformation only a bounded number of times. Eventually, we obtain a tile $T_0 = (G'', (v_1, \dots, v_t), (u_1, \dots, u_t))$ of width $t \leq k$ such that every edge-cut between the added vertices a and b has size at least t . By Menger's theorem, there exist pairwise edge-disjoint paths P_1, \dots, P_t in G'' and a permutation $\pi: [t] \rightarrow [t]$ such that for $1 \leq i \leq t$, the endvertices of P_i are v_i and $u_{\pi(i)}$. As mentioned above, we also have $\odot(T_0^n) = \odot(T^n)$ for every $n \geq 1$. \square

Observe that if T_1 and T_2 are weakly linked tiles and π_1 and π_2 are the corresponding permutations, then T_1T_2 is weakly linked for the permutation $\pi_2 \circ \pi_1$.

Proposition 3. *Let T be a weakly linked tile and let π be the permutation from the definition of weakly linked tiles. Let m be the least common multiple of the lengths of the cycles of the permutation π . Then, the tile T^m is linked.*

Proof. As we observed, the tile T^m is weakly linked for the permutation π^m , which is the identity permutation. Consequently, T^m is linked. \square

The following proposition shows that the components of linked tiles are themselves tiles (this is not necessarily the case in general; e.g., it is possible that a component contains different numbers of vertices from A and B).

Proposition 4. *Let $T = (G, A, B)$ be a linked tile, where G is the disjoint union of graphs G_1 and G_2 . Let A_1 and A_2 be the sequences of vertices of A in G_1 and G_2 , respectively, in the same order as in A . Let B_1 and B_2 be the sequences of vertices of B in G_1 and G_2 , respectively, in the same order as in B . Then $T_1 = (G_1, A_1, B_1)$ and $T_2 = (G_2, A_2, B_2)$ are linked tiles and for every $n \geq 1$, $\odot(T^n)$ is the disjoint union of $\odot(T_1^n)$ and $\odot(T_2^n)$.*

Proof. Let k be the width of T and consider some $i \in \{1, \dots, k\}$. Let a_i and b_i be the i -th vertices of A and B , respectively, and let P_i be the path from the definition of linkedness joining a_i with b_i . Note that either $P_i \subseteq G_1$ or $P_i \subseteq G_2$, and in particular a_i belongs to G_1 if and only if b_i belongs to G_1 . We conclude that T_1 and T_2 are linked tiles. Furthermore, the edges between the same vertices are added in the constructions of $\odot(T^n)$ and of $\odot(T_1^n)$ and $\odot(T_2^n)$, and thus $\odot(T^n) = \odot(T_1^n) \cup \odot(T_2^n)$. \square

Let $T = (G, A, B)$ be a tile of width k . Let H be a graph with vertices v_1, \dots, v_k and no edges, and let Z be the tile $(H, (v_1, \dots, v_k), (v_1, \dots, v_k))$. Let Z' be a copy of the tile Z with vertices v'_1, \dots, v'_k . A *tile drawing* of T is a drawing of ZTZ' in a closed disk such that the vertices $v_1, \dots, v_k, v'_k, v'_{k-1}, \dots, v'_1$ are drawn in the boundary of the disk in order.

We define $M(T) = \binom{|E(G)|+2k}{2}$. Note that there exists a tile drawing of T such that any two edges cross at most once; hence, this drawing has at most $M(T)$ crossings. By connecting n such tile drawings into a cycle, we conclude that $c_n(T) \leq M(T)n$ for every $n \geq 1$. We often use variants of the following useful observation (an analogous result with $s = 1$ was proved by Pinontoan and Richter [5]).

Lemma 5. *Let T be a tile of width k and let $s \geq 1$ an integer such that T^s is connected. For every $n \geq s + 1$, there exists a tile drawing of T^n with at most $c_n(T) + (8k + 1)M(T)s + \binom{2k}{2}$ crossings.*

Proof. Let \mathcal{G} be a drawing of $\odot(T^n)$ with $c_n(T)$ crossings. For $1 \leq i \leq n$, let a_i denote the number of crossings involving edges incident with the vertices of the i -th copy of T in \mathcal{G} , and let $a'_i = \sum_{j=i}^{i+s-1} a_j$, where $a_{n+x} = a_x$ for $x \geq 1$. Note that every crossing contributes 1 to at most four of the numbers a_i , and

thus $\sum_{i=1}^n a_i \leq 4c_n(T) \leq 4M(T)n$ and $\sum_{i=1}^n a'_i \leq 4M(T)ns$. Hence, without loss of generality we have $a'_1 \leq 4M(T)s$.

Let S_1 be the set of external edges of \mathcal{G} drawn between the first and the last copy of T , and let S_2 be the set of external edges of \mathcal{G} drawn between the s -th and the $(s+1)$ -th copy of T . Let v be any vertex of the first tile, and for each $e \in S_1 \cup S_2$, let P_e be a path starting with e and ending in v contained in the first s copies of T (which exists since T^s is connected).

Let \mathcal{G}_1 be the drawing of T^{n-s} obtained from \mathcal{G} by removing the first s copies of T . The drawing \mathcal{G}_1 has at most $c_n(T)$ crossings. Let \mathcal{G}_2 be the drawing obtained from \mathcal{G}_1 by adding back the vertex v and for each $e \in S_1 \cup S_2$, adding an edge e' between v and the endvertex of e contained on \mathcal{G}_1 , such that e' is drawn along the path P_e (perturbed slightly to avoid edges intersecting in infinite number of points, or three edges intersecting in one point). Note that for each $e \in S_1 \cup S_2$, every intersection of e' with an edge f of \mathcal{G}_1 appears next to an intersection of f with P_e . Consequently, e' intersects \mathcal{G}_1 in at most a'_1 points.

By splitting the vertex v into $2k$ vertices of degree one and shifting the vertices slightly, we can transform \mathcal{G}_2 into a tile drawing \mathcal{G}_3 of T^{n-s} that extends \mathcal{G}_1 , without creating any new intersections with \mathcal{G}_1 . Let $S' = \{e' : e \in S_1 \cup S_2\}$. If two edges of S' intersect more than once in the tile drawing \mathcal{G}_3 , we can eliminate the two crossings by swapping the parts of the edges between these crossings and shifting the edges at the crossings slightly (this does not affect the crossings with other edges). Furthermore, we can eliminate the crossings of edges of S' with themselves by removing parts of the edges. This way, we transform \mathcal{G}_3 into a tile drawing \mathcal{G}_4 such that no edge of S' intersects itself and each two edges of S' intersect at most once. Hence, \mathcal{G}_4 has at most $c_n(T) + 2ka'_1 + \binom{2k}{2} \leq c_n(T) + 8kM(T)s + \binom{2k}{2}$ crossings.

By combining \mathcal{G}_4 with s tile drawings of T (each with at most $M(T)$ crossings), we obtain the required tile drawing of T^n with at most $c_n(T) + (8k+1)M(T)s + \binom{2k}{2}$ crossings. \square

For an integer $n \geq 1$, let $t_n(T)$ denote the minimum number of crossings in a tile drawing of T^n . Pinontoan and Richter [5] observed that $t_n(T)$ is subadditive (i.e., for every $n_1, n_2 \geq 1$, we have $t_{n_1+n_2}(T) \leq t_{n_1}(T) + t_{n_2}(T)$), and by Fekete's subadditive lemma, the limit $\lim_{n \rightarrow \infty} t_n(T)/n$ exists. Note that a tile drawing of T^n can be turned into a drawing of $\odot(T^n)$ with the same number of crossings by identifying the corresponding vertices in the boundary of the drawing and suppressing the resulting vertices of degree two.

Hence, we have $c_n(T) \leq t_n(T)$, and Lemma 5 gives a rough converse. Hence, we obtain the following.

Corollary 6. *Let T be a tile. If there exists an integer $s \geq 1$ such that T^s is connected, then the limit $c(T) = \lim_{n \rightarrow \infty} c_n(T)/n$ exists and is equal to $\lim_{n \rightarrow \infty} t_n(T)/n$.*

A simple consequence of Lemma 5 is the following relationship between $c_n(T)/n$ and $c_m(T)/m$ for $m \gg n$.

Lemma 7. *Let $T = (G, A, B)$ be a connected tile of width k and let $\varepsilon > 0$ be a real number. Let $n_2 = 2 \left((8k+1)M(T) + \binom{2k}{2} \right) / \varepsilon$ and $a_0 = 2M(T)/\varepsilon$. If $n \geq n_2$ and $m \geq a_0 n$, then $c_m(T)/m \leq c_n(T)/n + \varepsilon$.*

Proof. By Lemma 5 applied with $s = 1$, there exists a tile drawing \mathcal{G}_1 of T^n with at most $c_n(T) + (8k+1)M(T) + \binom{2k}{2}$ crossings. Let \mathcal{G}_2 be a tile drawing of T with at most $M(T)$ crossings. Suppose that $m = an + b$, where a, b are nonnegative integers and $b \leq n - 1$. Let \mathcal{G} be the drawing of $\odot(T^m)$ obtained by combining a copies of \mathcal{G}_1 and b copies of \mathcal{G}_2 . Then, \mathcal{G} has at most $a(c_n(T) + (8k+1)M(T) + \binom{2k}{2}) + bM(T) \leq \frac{m}{n}c_n(T) + \frac{m}{n}((8k+1)M(T) + \binom{2k}{2}) + nM(T)$ crossings. Consequently,

$$\frac{c_m(T)}{m} \leq \frac{c_n(T)}{n} + \frac{1}{n} \left((8k+1)M(T) + \binom{2k}{2} \right) + \frac{n}{m}M(T) \leq \frac{c_n(T)}{n} + \varepsilon.$$

□

Corollary 8. *Let T be a connected tile and let $\varepsilon > 0$ be a real number. Let n_2 be as in Lemma 7. If $n \geq n_2$, then $c(T) \leq c_n(T)/n + \varepsilon$.*

Before we proceed with the proof of the main theorem, let us argue that the limit $c(T)$ exists even if T is not connected.

Lemma 9. *For every tile $T = (G, A, B)$ of width k , the limit $c(T) = \lim_{n \rightarrow \infty} c_n(T)/n$ exists. Furthermore, there exist integers $m \leq k!$ and $r \leq m|V(G)|$ and connected linked tiles T_1, \dots, T_r with at most $m|V(G)|$ vertices such that $c(T) = (c(T_1) + \dots + c(T_r))/m$.*

Proof. By Proposition 2, we can assume that T is weakly linked. Let m be as in Proposition 3 and let T_1, \dots, T_r be the maximal connected subtiles

of T^m . By Proposition 4, these subtiles are linked. Furthermore, the limits $c(T_1), \dots, c(T_r)$ exist by Corollary 6.

For $1 \leq i \leq r$, let T'_i be the subtile of T_i contained in the first copy of T in T^m . There exists a permutation π of $[r]$ such that $T_i = T'_i T'_{\pi(i)} T'_{\pi^2(i)} \dots T'_{\pi^{m-1}(i)}$. Let t be the number of cycles of π , and for $1 \leq i \leq t$, let ℓ_i be the length of the i -th cycle of π , let b_i an arbitrary element of the i -th cycle of π , and let $S_i = T'_{b_i} T'_{\pi(b_i)} T'_{\pi^2(b_i)} \dots T'_{\pi^{\ell_i-1}(b_i)}$. Then, for every $n \geq m$, $\odot(T^n)$ is the disjoint union consisting of $\gcd(n, \ell_i)$ copies of $\odot(S_i^{n/\gcd(n, \ell_i)})$ for $i = 1, \dots, t$. Note that $S_i^{m/\ell_i} = T_{b_i}$ is connected. Hence, by Corollary 6, the limit $c(S_i)$ exists.

Consider any $\varepsilon > 0$, and let $n_0 \geq m$ be large enough that $|c_n(S_i)/n - c(S_i)| \leq \varepsilon/t$ for every $n \geq n_0$ and $1 \leq i \leq t$. Let us remark that ℓ_i divides m , and thus $\ell_i \leq m$. Hence, for any $n \geq n_0 m$, we have

$$\begin{aligned} \left| \frac{c_n(T)}{n} - \sum_{i=1}^t c(S_i) \right| &= \left| \sum_{i=1}^t \frac{\gcd(n, \ell_i)}{n} c_{n/\gcd(n, \ell_i)}(S_i) - c(S_i) \right| \\ &\leq \sum_{i=1}^t \left| \frac{c_{n/\gcd(n, \ell_i)}(S_i)}{n/\gcd(n, \ell_i)} - c(S_i) \right| \\ &\leq \varepsilon. \end{aligned}$$

We conclude that $c(T) = \lim_{n \rightarrow \infty} c_n(T)/n$ exists (and is equal to $\sum_{i=1}^t c(S_i)$). Clearly, $c(T) = c(T^m)/m$, and by Proposition 4, we have $c(T^m) = \sum_{i=1}^r c(T_i)$. Hence, $c(T) = (c(T_1) + \dots + c(T_r))/m$ as required. \square

3 Drawings of periodic graphs

Let $T = (G, A, B)$ be a tile and n an integer. Consider a drawing \mathcal{G} of $\odot(T^n)$ and let $\beta > 0$ be a real number. The *weight* of a crossing between two internal edges is $1 + 2\beta$, between an internal and an external edge is $1 + \beta$ and between two external edges is 1. Let $\text{cr}_\beta(\mathcal{G})$ be the sum of the weights of the crossings in the drawing \mathcal{G} . The reason for introducing this weighted crossing number is the following lemma.

Lemma 10. *Let $T = (G, A, B)$ be a linked tile of width k , let n be an integer and let $\alpha \geq 0$, Q and $\beta > 0$ be real numbers. Let $n \geq 2$ be an integer and let \mathcal{G} be a drawing of $\odot(T^n)$ such that $\text{cr}_\beta(\mathcal{G}) \leq \alpha n + Q$. If the internal edges of*

some copy of T in the drawing \mathcal{G} participate in at least $\frac{1}{\beta} \binom{k}{2} + \alpha$ crossings, then there exists a drawing \mathcal{G}' of $\odot(T^{n-1})$ with $\text{cr}_\beta(\mathcal{G}') \leq \alpha(n-1) + Q$.

Proof. Let c be the number of crossings in \mathcal{G} involving the internal edges of a copy T' of T in \mathcal{G} , and suppose that $c \geq \frac{1}{\beta} \binom{k}{2} + \alpha$. Let P_1, \dots, P_k be the drawings of the paths from the definition of the linkedness of T' according to \mathcal{G} . Let \mathcal{G}'_0 be the drawing of $\odot(T^{n-1})$ obtained from \mathcal{G} by removing T' and by connecting the corresponding external edges incident with T' along the paths P_1, \dots, P_k . In case that more than two of the paths intersect in a vertex of the underlying graph of T' , we shift the new edges slightly so that at most two edges intersect in each point. Let S be the set of the resulting edges. Note that each of the crossings with the internal edges of T' in \mathcal{G} either disappears or becomes a crossing with an external edge of S in \mathcal{G}'_0 , and thus its weight is decreased at least by β . On the other hand, by drawing the edges along the paths P_1, \dots, P_k , we could introduce crossings between the edges of S ; let s be the number of these new crossings. This shows that $\text{cr}_\beta(\mathcal{G}'_0) \leq \text{cr}_\beta(\mathcal{G}) - \beta c + s$.

If two edges of S intersect more than once, we can eliminate the two crossings by swapping the parts of the edges between these crossings and shifting the edges at the crossings slightly (this does not affect the crossings with other edges). Furthermore, we can eliminate the crossings of edges of S with themselves by removing parts of the edges. In this way, we transform the drawing \mathcal{G}'_0 to a drawing \mathcal{G}' , where the number of crossings among the edges of S is at most $\binom{k}{2}$. Thus, we have $\text{cr}_\beta(\mathcal{G}') \leq \text{cr}_\beta(\mathcal{G}) - \beta c + \binom{k}{2} \leq \text{cr}_\beta(\mathcal{G}) - \alpha \leq \alpha(n-1) + Q$, as required. \square

For a tile T and two edges e_1 and e_2 of $\odot(T^n)$, the *cyclic tile distance* between e_1 and e_2 is the minimum number of distinct tiles that a path in $\odot(T^n)$ between the two edges must intersect.

Lemma 11. *Let $T = (G, A, B)$ be a connected linked tile of width k and let $\beta, \varepsilon > 0$ and $\alpha \geq 0$ be real numbers. Let $c = \frac{1}{\beta} \binom{k}{2} + \alpha$ and $Q_0 = 2k(2|E(G)| + 2c + 4k)(1 + \beta) + 4k^2 + 2\alpha$. Let $n_1 \geq n_0 \geq 1$ be integers and let $Q \geq Q_0$ be a real number, and suppose that n is the smallest integer such that $n \geq n_1$ and there exists a drawing \mathcal{G} of $\odot(T^n)$ with $\text{cr}_\beta(\mathcal{G}) \leq \alpha n + Q$. Furthermore, assume that*

- \mathcal{G} is chosen among the drawings of $\odot(T^n)$ so that $\text{cr}_\beta(\mathcal{G})$ is the smallest possible, and

- for $n_0 \leq m < n_1$, there is no drawing \mathcal{G}' of $\odot(T^m)$ with $\text{cr}_\beta(\mathcal{G}') \leq \alpha m + Q_0$.

If $n \geq 2n_1 + 2$, then the cyclic tile distance between any two crossing edges of \mathcal{G} is at most $n_0 + 1$.

Proof. Since $n > n_1$, the minimality of n and Lemma 10 imply that every copy of G in \mathcal{G} is intersected at most c times.

Suppose that \mathcal{G} contains two crossing edges e_1 and e_2 with cyclic tile distance at least $n_0 + 2$. Let \mathcal{G}_1 and \mathcal{G}_2 be the subdrawings of \mathcal{G} consisting of the tiles on the two paths between e_1 and e_2 in the cycle forming $\odot(T^n)$; in case that e_1 or e_2 is an internal edge of a tile, this tile is included neither in \mathcal{G}_1 nor in \mathcal{G}_2 . For $i \in \{1, 2\}$, \mathcal{G}_i is a drawing of T^{k_i} , where k_1 and k_2 are integers such that $k_1 + k_2 = n$ if both e_1 and e_2 are external, $k_1 + k_2 = n - 1$ if one of e_1 and e_2 is internal and $k_1 + k_2 = n - 2$ if both e_1 and e_2 are internal. Furthermore, we have $k_1, k_2 \geq n_0$. Since $n \geq 2n_1 + 2$, we can also assume that $k_2 \geq n_1$.

Suppose that e_1 is an external edge, and let S be the set of edges between the two copies of G that are joined by e_1 . Let x be the sum of weights of crossings of e_1 in \mathcal{G} . Consider an edge $e' \in S$ and let y be the sum of weights of crossings of e' in \mathcal{G} . Since T is connected, we can redraw e' to follow a path in the tiles with which it is incident to come close to the ends of e_1 and then follow the drawing of e_1 instead of its current drawing. Since every copy of G is intersected at most c times, the crossings on the redrawn edge e' would have weight at most $(2|E(G)| + 2c + 4k)(1 + \beta) + x$. By the minimality of $\text{cr}_\beta(\mathcal{G})$, we have $(2|E(G)| + 2c + 4k)(1 + \beta) + x \geq y$, and by symmetry, $|x - y| \leq (2|E(G)| + 2c + 4k)(1 + \beta)$. Therefore, we can redraw all edges of S along e_1 and increase $\text{cr}_\beta(\mathcal{G})$ by at most $k(2|E(G)| + 2c + 4k)(1 + \beta)$. If e_2 is external, we perform a similar transformation for e_2 as well; this may incur an additional penalty of at most $4k^2$ for the intersections between the rerouted edges. By allowing this penalty, we can achieve any specified order of the edges in S close to the crossing of e_1 with e_2 . In case that e_1 or e_2 are internal, we can perform the same transformation after first eliminating the copies of G containing e_1 or e_2 similarly to the proof of Lemma 10.

Finally, we can match the rerouted edges in the vicinity of the crossing of e_1 and e_2 and obtain drawings \mathcal{G}'_1 and \mathcal{G}'_2 of $\odot(T^{k_1})$ and $\odot(T^{k_2})$, respectively, such that $\text{cr}_\beta(\mathcal{G}'_1) + \text{cr}_\beta(\mathcal{G}'_2) \leq \text{cr}_\beta(\mathcal{G}) + 2k(2|E(G)| + 2c + 4k)(1 + \beta) + 4k^2 = \text{cr}_\beta(\mathcal{G}) + Q_0 - 2\alpha$. Since $n > k_2 \geq n_1$, the minimality of n implies that $\text{cr}_\beta(\mathcal{G}'_2) > \alpha k_2 + Q$. Furthermore, note that $\text{cr}_\beta(\mathcal{G}'_1) > \alpha k_1 + Q_0$ (this follows

from the assumptions of the lemma if $k_1 < n_1$, and by the minimality of n otherwise). Therefore, $\text{cr}_\beta(\mathcal{G}'_1) + \text{cr}_\beta(\mathcal{G}'_2) > \alpha(k_1 + k_2) + Q + Q_0 \geq \alpha n + Q + Q_0 - 2\alpha$. Therefore, we have $\text{cr}_\beta(\mathcal{G}) > \alpha n + Q$, which is a contradiction. \square

The main approximation result follows straightforwardly from the next lemma.

Lemma 12. *Let $T = (G, A, B)$ be a connected linked tile of width k and let $\varepsilon \leq 1$ be a positive real number. Let $\alpha = c(T) + \frac{1}{2}\varepsilon$, $\beta = \varepsilon/(8\alpha)$, $c = \frac{1}{\beta}(\binom{k}{2} + \alpha)$, $Q_0 = 2k(2|E(G)| + 2c + 4k)(1 + \beta) + 4k^2 + 2\alpha$, $n_0 = \lceil 2Q_0/\varepsilon \rceil$, $Q = 8c(n_0 + 1)(1 + \beta) + 4k^2(n_0 + 2)^2 + 2\binom{k}{2}$ and $n_1 = \lceil 2Q/\varepsilon \rceil$. Then, there exists n such that $n_0 \leq n \leq 2n_1 + 1$ and $c_n(T)/n \leq c(T) + \varepsilon$.*

Proof. Since $\lim_{t \rightarrow \infty} \frac{c_t(T)}{t} = c(T)$, for every sufficiently large t , there exists a drawing \mathcal{G} of $\odot(T^t)$ with $\text{cr}(\mathcal{G}) \leq (c(T) + \varepsilon/(4(1 + 2\beta)))t$. Note that $\text{cr}_\beta(\mathcal{G}) \leq (1 + 2\beta)\text{cr}(\mathcal{G})$, and thus $\text{cr}_\beta(\mathcal{G}) \leq (c(T) + 2\beta c(T) + \frac{1}{4}\varepsilon)t \leq \alpha t$ for such an integer t . Hence, there exists the smallest integer n such that $n \geq n_1$ and for some drawing \mathcal{G} of $\odot(T^n)$, we have $\text{cr}_\beta(\mathcal{G}) \leq \alpha n + Q$. Let \mathcal{G} be such a drawing with the smallest possible $\text{cr}_\beta(\mathcal{G})$.

If there exists m such that $n_0 \leq m < n_1$ and there is a drawing \mathcal{G}' of $\odot(T^m)$ with $\text{cr}_\beta(\mathcal{G}') \leq \alpha m + Q_0$, then the claim of the lemma holds, since $\text{cr}(\mathcal{G}') \leq \text{cr}_\beta(\mathcal{G}') \leq \alpha m + Q_0 = (c(T) + \frac{1}{2}\varepsilon + Q_0/m)m \leq (c(T) + \frac{1}{2}\varepsilon + Q_0/n_0)m \leq (c(T) + \varepsilon)m$. Therefore, assume that there is no such m .

Similarly, if $n \leq 2n_1 + 1$, then the claim holds, since $\text{cr}(\mathcal{G}) \leq \text{cr}_\beta(\mathcal{G}) \leq \alpha n + Q = (c(T) + \frac{1}{2}\varepsilon + Q/n)n \leq (c(T) + \frac{1}{2}\varepsilon + Q/n_1)n \leq (c(T) + \varepsilon)n$. Thus, we may assume that $n \geq 2n_1 + 2$ and we can apply Lemma 11 to conclude that any two crossing edges in \mathcal{G} have cyclic tile distance at most $n_0 + 1$.

Let k_1 and k_2 be integers such that $k_1 + k_2 = n$ and $k_1, k_2 \geq n_1$. The drawing \mathcal{G} can be decomposed to drawings \mathcal{G}_1 and \mathcal{G}_2 of T^{k_1} and T^{k_2} . Consider the drawing \mathcal{G}_1 , and let A_0 and B_0 be the vertices of the first and the last tile of \mathcal{G}_1 , respectively, incident with the external edges of $\mathcal{G} \setminus \mathcal{G}_1$. Let S be the set of edges of $\mathcal{G} \setminus \mathcal{G}_1$ incident with $A_0 \cup B_0$. Let Z be the set of edges of \mathcal{G}_1 that are at cyclic tile distance at least $n_0 + 2$ from the edges of S and let R be the edges of \mathcal{G}_1 at cyclic distance at most $n_0 + 1$ from S . Since T is linked, there exist pairwise edge-disjoint paths P_1, \dots, P_k in $\mathcal{G}_2 \cup S$ joining the corresponding vertices of A_0 and B_0 . For $i = 1, \dots, k$, let us add an edge to \mathcal{G}_1 between the i -th vertices of A_0 and B_0 drawn along P_i ; let \mathcal{G}'_1 be the resulting drawing and let S' be the set of newly added edges.

By Lemma 11, the edges of S' do not intersect Z . Let us estimate the weight of crossings between S' and R . Note that each such crossing corresponds to a crossing in \mathcal{G} between an edge of R and of \mathcal{G}_2 . The internal edges of R belong to at most $2(n_0 + 1)$ distinct tiles of \mathcal{G} , and thus by Lemma 10, they are intersected at most $2c(n_0 + 1)$ times. Consider the external edges of R . By Lemma 11, they can only intersect the edges of \mathcal{G}_2 at cyclic tile distance at most $n_0 + 1$ from S . By Lemma 10, it follows that there are at most $2c(n_0 + 1)$ intersections between external edges of R and internal edges of \mathcal{G}_2 . Finally, note that each two external edges of \mathcal{G} intersect at most once, as otherwise we could decrease $\text{cr}_\beta(\mathcal{G})$. Therefore, the number of crossings between external edges of R and \mathcal{G}_2 is bounded by $2k^2(n_0 + 2)^2$.

As usual, we can assume that each two edges of S' intersect at most once in \mathcal{G}'_1 by redrawing them if necessary. We conclude that $\text{cr}_\beta(\mathcal{G}'_1) \leq \text{cr}_\beta(\mathcal{G}_1) + 4c(n_0 + 1)(1 + \beta) + 2k^2(n_0 + 2)^2 + \binom{k}{2} = \text{cr}_\beta(\mathcal{G}_1) + \frac{1}{2}Q$. Similarly, we can obtain a drawing \mathcal{G}'_2 of $\odot(T^{k_2})$ with $\text{cr}_\beta(\mathcal{G}'_2) \leq \text{cr}_\beta(\mathcal{G}_2) + \frac{1}{2}Q$. We have $\text{cr}_\beta(\mathcal{G}_1) + \text{cr}_\beta(\mathcal{G}_2) \leq \text{cr}_\beta(\mathcal{G})$, and thus $\text{cr}_\beta(\mathcal{G}'_1) + \text{cr}_\beta(\mathcal{G}'_2) \leq \text{cr}_\beta(\mathcal{G}) + Q$.

However, since $n_1 \leq k_1, k_2 < n$, the minimality of n implies that $\text{cr}_\beta(\mathcal{G}'_1) > \alpha k_1 + Q$ and $\text{cr}_\beta(\mathcal{G}'_2) > \alpha k_2 + Q$. Therefore, $\text{cr}_\beta(\mathcal{G}) > \alpha n + Q$, which is a contradiction. \square

We are ready to give the proof of the main theorem.

Proof of Theorem 1. By Lemma 9, we can assume that the tile $T = (G, A, B)$ is connected and linked (otherwise, we consider each connected subtile of T^m separately). Let k be the width of T . Let ε_1 be a constant to be chosen later and let us consider the following quantities:

- $\alpha_d = \frac{1}{2}\varepsilon_1$ and $\alpha_u = M(T) + \frac{1}{2}\varepsilon_1$
- $\beta_d = \varepsilon_1/(8\alpha_u)$ and $\beta_u = \varepsilon_1/(8\alpha_d)$
- $c_d = \frac{1}{\beta_u}(\binom{k}{2} + \alpha_d)$ and $c_u = \frac{1}{\beta_d}(\binom{k}{2} + \alpha_u)$
- $Q_{0,d} = 2k(2|E(G)| + 2c_d + 4k)(1 + \beta_d) + 4k^2 + 2\alpha_d$ and $Q_{0,u} = 2k(2|E(G)| + 2c_u + 4k)(1 + \beta_u) + 4k^2 + 2\alpha_u$
- $n_{0,d} = \lceil 2Q_{0,d}/\varepsilon_1 \rceil$ and $n_{0,u} = \lceil 2Q_{0,u}/\varepsilon_1 \rceil$
- $Q_u = 8c_u(n_{0,u} + 1)(1 + \beta_u) + 4k^2(n_{0,u} + 2)^2 + 2\binom{k}{2}$ and
- $n_{1,u} = \lceil 2Q_u/\varepsilon_1 \rceil$.

These numbers are chosen in such a way that they are related to the values used in Lemma 12. More precisely, $\alpha_d, \beta_d, c_d, \dots$ give lower bounds and $\alpha_u, \beta_u, c_u, \dots$ give upper bounds on the corresponding quantities in Lemma 12, used with ε_1 in the role of ε . In particular, if n_0 and n_1 are the constants of Lemma 12 for T and ε_1 , then $n_0 \geq n_{0,d}$ and $n_1 \leq n_{1,u}$. Furthermore, $n_{0,d} = \Theta(1/\varepsilon_1)$ and $n_{1,u} = \Theta(1/\varepsilon_1^5)$ are computable, given T . (Here and below, all constants involved in the Θ -notation depend only on T .)

Let $n_2 = \Theta(1/\varepsilon)$ and $a_0 = \Theta(1/\varepsilon)$ be the constants of Lemma 7 applied for $\varepsilon/2$. Let us now choose $\varepsilon_1 \leq \varepsilon/2$ in such a way that $n_{0,d} \geq n_2$ and $\varepsilon_1 = \Theta(\varepsilon)$. Let $N = a_0 n_{1,u}$ and note that $N = \Theta(1/\varepsilon^6)$.

By Lemma 12, there exists n such that $n_{0,d} \leq n \leq n_{1,u}$ and $c_n(T)/n \leq c(T) + \varepsilon_1 \leq c(T) + \varepsilon/2$. By Lemma 7, we have $c_t(T)/t \leq c_n(T)/n + \varepsilon/2 \leq c(T) + \varepsilon$ for every $t \geq N$. Conversely, Corollary 8 implies that $c_t(T)/t \geq c(T) - \varepsilon$. \square

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